Non-standard holomorphic structures of line bundles on the quantum projective space

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Based on arXiv:2510.14263

A joint work with Graveman, La Rue, MacArthur, and Pesin

GaGA Seminar

Texas Christian University

November 2, 2026

Acknowledgments

This research was supported by the National Science Foundation under Grant No. DMS-2243991, and took place at an REU hosted by East Texas A&M University, USA.





Outline

1. A quick introduction to noncommutative complex differential geometry
2. A quick review of $\mathbb{C}P^1_q$ and line bundles on it
3. Gauge equivalences of antiholomorphic connections
4. Existence of non-standard antiholomorphic connections

Complex manifolds

- A complex manifold is a manifold with a complex structure, that is an atlas of charts to the open unit ball in the complex coordinate space $\mathbb{C}^n$, such that the transition maps are holomorphic.
- An **almost complex structure** on a (real) manifold X is a linear map $J : TX \to TX$ such that $J^2 = -\mathrm{id}$.
- $TX \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into $TX^{1,0}$ and $TX^{0,1}$, the i and $-i$ eigenspaces of J.
- We have the decomposition $\Omega^n(X, \mathbb{C}) = \oplus_{p+q=n} \Omega^{p,q}(X)$.
- We can define $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$.
- Every complex manifold has a canonical almost complex structure.
- An almost complex structure J is called **integrable** if it comes from a complex structure.

Theorem (Newlander-Nirenberg theorem)

An almost complex structure J is integrable if and only if $\overline{\partial}^2 = 0$.

Holomorphic vector bundles

Theorem (Serre-Swan theorem)

Complex C^∞-vector bundles over a C^∞ manifold $X \overset{1-1}{\longleftrightarrow}$ Finite rank projective modules over the algebra $C^\infty(X)$.

- An antiholomorphic connection on a C^∞-vector complex vector bundle E over a complex manifold X is a map

$$\overline{\nabla}_E : C^\infty(X, E) \to \Omega^{0,1}(X, E)$$

such that $\overline{\nabla}_E(fs) = \overline{\partial}(f)s + f\overline{\nabla}_E(s)$.

Theorem (Koszul-Malgrange theorem)

Holomorphic vector bundles over a complex manifold $\overset{1-1}{\longleftrightarrow}$ complex C^∞-vector bundles E equipped with an antiholomorphic connection $\overline{\nabla}_E$ such that $(\overline{\nabla}_E)^2 = 0$.

Differential $*$-calculi

- A $*$-algebra is an algebra A over $\mathbb{C}$ together with an $\mathbb{R}$-linear map $* : A \to A$ satisfying the following conditions:
  - $(ab)^* = b^* a^*$;
  - $(*)^2 = \mathrm{id}$;
  - $1^* = 1$ if A is unital;
  - $(ca)^* = \bar{c} a^*$ for $c \in \mathbb{C}$.
- A **differential $*$-calculus** over a $*$-algebra A is a triple $(\Omega^\bullet(A), \mathrm{d}, *)$ such that
  - $(\Omega^\bullet(A), \mathrm{d})$ is a differential graded algebra such that

$$\Omega^k(A) = \mathrm{span}\{a_0 \mathrm{d}a_1 \dots \mathrm{d}a_k | a_0, \dots, a_k \in A\};$$

  - $(*)^2 = \mathrm{id}$, $*$ preserves the grading, $(\mathrm{d}(\xi))^* = \mathrm{d}(\xi^*)$, $(\xi\eta)^* = (-1)^{|\xi||\eta|} \eta^* \xi^*$.

Noncommutative complex manifolds

We follow [BPS13]

- An **almost complex structure** on a differential $*$-calculus $(\Omega^\bullet(A), \mathrm{d}, *)$ is a degree zero derivation $J : \Omega^\bullet(A) \to \Omega^\bullet(A)$ such that $J|_A = 0$, $J^2|_{\Omega^1(A)} = -\mathrm{id}$, and $(J\xi)^* = J(\xi^*)$ for any $\xi \in \Omega^1(A)$.
- We have the decomposition $\Omega^n(A) = \oplus_{p+q=n}\Omega^{p,q}(A)$, where $\Omega^{p,q}(A) := \{\xi \in \Omega^{p+q}(A) | J\xi = (p-q)i\xi\}$.
- We can define $\partial : \Omega^{p,q}(A) \to \Omega^{p+1,q}(A)$ and $\overline{\partial} : \Omega^{p,q}(A) \to \Omega^{p,q+1}(A)$.
- We call J **integrable** if $\overline{\partial}^2 = 0$.
- The quadruple $(\Omega^\bullet(A), \mathrm{d}, *, J)$ where J is integrable defines a **noncommutative complex manifold**.

Quantum projective spaces

- An interesting example of noncomutative complex manifold is the quantum projective space $\mathbb{C}P_q^n$.
- In the commutative case we have $\mathbb{C}P^n \cong SU(n+1)/U(n)$.
- In the noncommutative case we replace $SU(n+1)$ and $U(n)$ by the corresponding **quantum groups**

$$\mathbb{C}P_q^n \cong "SU_q(n+1)/U_q(n)" \text{ (will be explained later).}$$

- In this talk we focus on $\mathbb{C}P_q^1$. But most constructions can be easily generalized to the n-dimensional case.
- Reference: [KLvS11] for $\mathbb{C}P_q^1$, [DL13] for $\mathbb{C}P_q^n$.

$\mathcal{A}(SU_q(2))$

- In this talk we fix a real parameter $0 < q < 1$.
- The defining matrix
$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$$
- Relations are given by
$$UU^* = U^*U = \mathrm{id}.$$
- $\mathcal{A}(SU_q(2))$ is the $*$-algebra generated by entries of U.
- We consider $\mathcal{A}(SU_q(2))$ as the **coordinate algebra** of $SU_q(2)$.

$\mathcal{A}(SU_q(2))$, cont'd

- More concretely, $\mathcal{A}(SU_q(2))$ is the unital $*$-algebra generated by a and c with relations

$$ac = qca,\ ac^* = qc^*a,\ cc^* = c^*c,$$

$$a^*a + c^*c = aa^* + q^2cc^* = 1.$$

- $* \circ * = \mathrm{id}$, $(xy)^* = y^*x^*$.

The algebra $\mathcal{A}(\mathbb{C}P^1_q)$

- $\mathcal{A}(\mathbb{C}P^1_q)$ is a $*$-subalgebra of $\mathcal{A}(SU_q(2))$.
- A set of generators of $\mathcal{A}(\mathbb{C}P^1_q)$ is

$$B_- = ac^*,\ B_+ = ca^*,\ B_0 = cc^*.$$

- Relations

$$B_-B_0 = q^2B_0B_-,\ B_+B_0 = q^{-2}B_0B_+,$$

$$B_-B_+ = q^2B_0(1-q^2B_0),\ B_+B_- = B_0(1-B_0),$$

$$(B_+)^* = B_-,\ (B_0)^* = B_0.$$

- If $q = 1$ and consider $\mathbb{C}P^1$ as the standard sphere in the xyz-coordinate system, i.e. $\mathbb{C}P^1 \cong S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$, then

$$B_0 \leftrightsquigarrow \frac{1-z}{2},\ B_+ \leftrightsquigarrow \frac{x+iy}{2},\ B_- \leftrightsquigarrow \frac{x-iy}{2}.$$

- We can equip $\mathcal{A}(\mathbb{C}P^1_q)$ with a differential $*$-calculus plus an integrable almost complex structure ([KLvS11]).
- Hence $\mathbb{C}P^1_q$ is a noncommutative complex manifold.

The C^*-completion $\mathcal{C}(\mathbb{C}P^1_q)$ and its faithful Representation

- $\mathcal{A}(\mathbb{C}P^1_q)$ itself is not a C^*-algebra.
- But we can look into the C^***-completion** $\mathcal{C}(\mathbb{C}P^1_q)$.
- [AK18]: Let $\ell^2 = \{(z_0, \ldots, z_n, \ldots) | \sum_{i=1}^{\infty} |z_i|^2 < \infty\}$. We have a faithful representation of $\pi : \mathcal{C}(\mathbb{C}P^1_q) \to B(\ell^2)$ by the following operators,
  - $(\pi B_0)(e_n) = q^{2n} e_n$
  - $(\pi B_+)(e_n) = q^{n+1}\sqrt{1-q^{2n+2}} e_{n+1}$
  - $(\pi B_-)(e_n) = q^{n}\sqrt{1-q^{2n}} e_{n-1}$
- Consequence: $\|B_0\| = 1$ and $sp(B_0) = \{1, q^2, q^4, \ldots\} \cup \{0\}$
- In particular, 0 is the only cluster point of $sp(B_0)$.

Line bundles $\mathcal{L}_n$ on $\mathbb{C}P^1_q$

We follow [KLvS11].

- Let $\mathcal{L}_n$ be the subset of $\mathcal{A}(SU_q(2))$ defined by
$$\mathcal{L}_n := \{x \in \mathcal{A}(SU_q(2)) : K \triangleright x = q^{\frac{n}{2}} x\}.$$
- $\mathcal{L}_0 = \mathcal{A}(\mathbb{C}P^1_q)$, $\mathcal{L}_n^* = \mathcal{L}_{-n}$, $\mathcal{L}_m \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \mathcal{L}_n \cong \mathcal{L}_{m+n}$.
- The $\mathcal{L}_n$'s are projective $\mathcal{A}(\mathbb{C}P^1_q)$-bimodules.
- As $\mathcal{A}(\mathbb{C}P^1_q)$-modules, the $\mathcal{L}_n$'s have generators
$$\{(c^*)^m (a^*)^{n-m},\ m = 0, \dots n\} \text{ for } n \geq 0;$$
$$\{c^m a^{-n-m},\ m = 0, \dots -n\} \text{ for } n \leq 0$$
- $\mathcal{L}_n$ is the noncommutative analogue of the line bundle $\mathcal{O}(-n)$ in algebraic geometry.

The $\overline{\partial}$-operator on $\mathbb{C}P^1_q$

We follow [KLvS11].

- Recall that we can equip $\mathcal{A}(\mathbb{C}P^1_q)$ with a differential $*$-calculus plus an integrable almost complex structure.
- We have $\Omega^{0,1}(\mathbb{C}P^1_q) \cong \mathcal{L}_{-2}$ as $\mathcal{A}(\mathbb{C}P^1_q)$-bimodules.
- Then the map $\overline{\partial} : \mathcal{A}(\mathbb{C}P^1_q) \to \Omega^{0,1}(\mathbb{C}P^1_q)$ is given by

  $$\overline{\partial}(B_0) = -q^{-\frac{1}{2}} ca,\ \overline{\partial}(B_+) = q^{\frac{1}{2}} c^2,\ \overline{\partial}(B_-) = -q^{-\frac{1}{2}} a^2.$$

  and extend by the Leibniz rule.
- $\overline{\partial}^2 = 0$ by dimension reason.
- We can extend $\overline{\partial}$ to a closed operator on $\mathcal{C}(\mathbb{C}P^1_q)$.

Holomorphic vector bundles on $\mathbb{C}P_q^1$

We follow [KLvS11].

- A **left holomorphic vector bundle** on $\mathbb{C}P_q^1$ is defined to be a finitely generated projective left $\mathcal{A}(\mathbb{C}P_q^1)$-module $\mathcal{E}$ together with a flat antiholomorphic connection
$$\overline{\nabla}_{\mathcal{E}} : \mathcal{E} \to \Omega^{0,1}(\mathbb{C}P_q^1) \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \mathcal{E}$$
which satisfies $\overline{\nabla}_{\mathcal{E}}(fe) = (\bar{\partial} f) \otimes e + f\overline{\nabla}_{\mathcal{E}}(e)$ for $f \in \mathcal{A}(\mathbb{C}P_q^1)$ and $e \in \mathcal{E}$.
A right holomorphic vector bundle is defined similarly.
- A **bimodule holomorphic structure** on a finitely generated projective $\mathcal{A}(\mathbb{C}P_q^1)$-bimodule $\mathcal{E}$ equipped with a flat left antiholomorphic connection $\overline{\nabla}_{\mathcal{E}}$ and a $\mathcal{A}(\mathbb{C}P_q^1)$-bimodule isomorphism
$$\Phi_{\mathcal{E}} : \mathcal{E} \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \Omega^{0,1}(\mathbb{C}P_q^1) \xrightarrow{\sim} \Omega^{0,1}(\mathbb{C}P_q^1) \otimes_{\mathcal{A}(\mathbb{C}P_q^1)} \mathcal{E}$$
such that $\overline{\nabla}_{\mathcal{E}}(ef) = (\overline{\nabla}_{\mathcal{E}} e)f + \Phi_{\mathcal{E}}(e \otimes \bar{\partial} f)$.

The standard holomorphic structure on $\mathcal{L}_n$'s

We follow [KLvS11].

- Define the **standard antiholomorphic connection** $\overline{\nabla}^{(n)} : \mathcal{L}_n \to \Omega^{0,1}(\mathbb{C}P^1_q) \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \mathcal{L}_n$ as
$$\overline{\nabla}^{(n)}(s) := q^{\frac{n}{2}} FK \triangleright s.$$
- For $f \in \mathcal{A}(\mathbb{C}P^1_q)$ and $s \in \mathcal{L}_n$ we have
$$\overline{\nabla}^{(n)}(fs) = \overline{\partial}(f)s + f\overline{\nabla}^{(n)}(s).$$
- $(\overline{\nabla}^{(n)})^2 = 0$ by dimension reason.
- There exists a bimodule isomorphism
$$\Phi_{(n)} : \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^{0,1}(\mathbb{C}P^1_q) \to \Omega^{0,1}(\mathbb{C}P^1_q) \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \mathcal{L}_n$$
such that $\overline{\nabla}^{(n)}$ is a bimodule holomorphic structure on $\mathcal{L}_n$.
- We can define tensor product of standard antiholomorphic connections on line bundles.

The Dolbeault cohomology of the $\mathcal{L}_n$'s

- We sometimes omit the superscript on $\overline{\nabla}^{(n)}$ when there is no confusion.
- [KLvS11]:

$$H^0_{\overline{\nabla}}(\mathbb{C}P^1_q, \mathcal{L}_{-n}) = \begin{cases} 0 & \text{if } n < 0 \\ \mathbb{C}^{n+1} & \text{if } n \geq 0 \end{cases};$$

- Moreover the space $\oplus_n H^0_{\overline{\nabla}}(\mathbb{C}P^1_q, \mathcal{L}_n)$ has a ring structure and is isomorphic to the quantum plane

$$\oplus_n H^0_{\overline{\nabla}}(\mathbb{C}P^1_q, \mathcal{L}_n) \simeq \mathbb{C}\langle a, c\rangle/(ac - qca).$$

- [DL13]: $H^1_{\overline{\nabla}}(\mathbb{C}P^1_q, \mathcal{L}_0) = 0$.
- The above results show the analogue between the noncommutative and commutative case.

Antiholomorphic connections on $\mathcal{L}_n$ and gauge equivalence

In the naive sense, the antiholomorphic connections on $\mathcal{L}_n$ is not unique. For any $\theta \in \Omega^{0,1}(\mathbb{C}P^1_q)$, we can define a new **left** antiholomorphic connection $\overline{\nabla}^{(n)}_\theta$ by

$$\overline{\nabla}^{(n)}_\theta(s) := \overline{\nabla}^{(n)}(s) - \Phi_{(n)}(s \otimes \theta).$$

Definition

We call two left antiholomorphic connections $\overline{\nabla}^{(n)}_{\theta_1}$ and $\overline{\nabla}^{(n)}_{\theta_2} : \mathcal{L}_n \to \Omega^{0,1}(\mathbb{C}P^1_q) \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \mathcal{L}_n$ **gauge equivalent** if there exists an invertible element $g \in \mathrm{End}_{\mathcal{A}(\mathbb{C}P^1_q)}\mathcal{L}_n$ such that

$$g^{-1} \circ \overline{\nabla}^{(n)}_{\theta_1} \circ g = \overline{\nabla}^{(n)}_{\theta_2}.$$

Question ([KLvS11], Page 872)

Is the antiholomorphic connection on $\mathcal{L}_n$ unique up to gauge equivalence?

More on gauge equivalence

- $\mathrm{End}_{\mathcal{A}(\mathbb{C}P^1_q)}\mathcal{L}_n \cong \mathcal{A}(\mathbb{C}P^1_q)$.
- We have $H^{0,1}_{\overline{\partial}}(\mathbb{C}P^1_q) = 0$, so we can find an $f \in \mathcal{A}(\mathbb{C}P^1_q)$ such that $\overline{\partial}(f) = \theta$.
- We consider the C^*-completion of $\mathcal{A}(\mathbb{C}P^1_q)$, $\mathcal{C}(\mathbb{C}P^1_q)$.

Question (Updated form)

For any $f \in \mathcal{A}(\mathbb{C}P^1_q)$, can we find an invertible $g \in \mathcal{C}^\times(\mathbb{C}P^1_q) \cap Dom(\overline{\partial})$ such that $g^{-1} \in Dom(\overline{\partial})$ and

$$\overline{\partial}(g) = g\overline{\partial}(f)?$$

Moreover for $f_1, f_2 \in \mathcal{A}(\mathbb{C}P^1_q)$, can we find an invertible $g \in \mathcal{C}^\times(\mathbb{C}P^1_q) \cap Dom(\overline{\partial})$ such that $g^{-1} \in Dom(\overline{\partial})$ and

$$\overline{\partial}(g) = g\overline{\partial}(f_1) - \overline{\partial}(f_2)g?$$

- We do not know whether $g \in \mathcal{C}^\times(\mathbb{C}P^1_q) \cap \mathrm{Dom}(\overline{\partial})$ implies $g^{-1} \in \mathrm{Dom}(\overline{\partial})$.
- Tensor product $\Rightarrow$ gauge equivalent classes of antiholomorphic connections on $\mathcal{L}_n$ and $\mathcal{L}_m$ are one-to-one correspondent.

The first attempt

- $\overline{\partial}(f^n) \neq nf^{n-1}\overline{\partial}(f)$, so simply taking $g = \exp(f) = 1 + f + \frac{f^2}{2} + \ldots$ does not help.
- For B_0 we can show that $(\overline{\partial}B_0)B_0 = q^2 B_0 \overline{\partial}B_0$. Therefore
$$\overline{\partial}B_0^n = \frac{1-q^{2n}}{1-q^2} B_0^{n-1}\overline{\partial}B_0 = (1 + q^2 + \ldots + q^{2n-2})B_0^{n-1}\overline{\partial}B_0.$$
- We define the q^2-integer $[n]_{q^2} := 1 + q^2 + \ldots + q^{2n-2}$, and the q^2-factorial $[n]!_{q^2} := \prod_{k=1}^{n}[k]_{q^2}$.
- Then for $f = kB_0$ where $k \in \mathbb{C}$, the q^2-exponential
$$g := \sum_{n=0}^{\infty} \frac{k^n B_0^n}{[n]!_{q^2}}$$
satisfies $\overline{\partial}(g) = g\overline{\partial}(f)$.
- Radius of convergence is $\frac{1}{1-q^2}$ instead of ∞.
- Recall $\|B_0\| = 1$. The series converges for $|k| < \frac{1}{1-q^2}$ and diverges for $|k| > \frac{1}{1-q^2}$.
- Divergence does not mean that the solution does not exist.

Some generalities on gauge equivalences

- The fact that $H^0_{\overline{\partial}}(\mathbb{C}P^1_q) = \mathbb{C}$ has the following important consequences:

Lemma

For a fixed f, if $\overline{\partial}(g) = g\overline{\partial}(f)$ has an invertible solution g, then any solution must be a complex scalar multiple of g.

Corollary

*For a fixed f, if $\overline{\partial}(g) = g\overline{\partial}(f)$ has a nonzero, non-invertible solution g, then it **cannot** have an invertible solution.*

- We do not have a similar result for solutions to $\overline{\partial}(g) = g\overline{\partial}(f_1) - \overline{\partial}(f_2)g$.

Functional Calculus with B_0

- Goal: solve

$$\overline{\partial}(g) = g\overline{\partial}(f)$$

in $C^*\{1, B_0\} \subset \mathcal{C}(\mathbb{C}P^1_q)$, the unital C^*-subalgebra generated by B_0.

- Let

$$\psi : C(sp(B_0)) \xrightarrow{\sim} C^*\{1, B_0\} \subset \mathcal{C}(\mathbb{C}P^1_q)$$

be the continuous functional calculus isomorphism.

- We will omit Ψ when there is no danger of confusion.
- Recall that we have $sp(B_0) = \{1, q^2, q^4, \ldots\} \cup \{0\}$.
- The only conditions that a function f on $sp(B_0)$ is continuous and invertible are:
  1. Continuous
     - $\lim_{k\to\infty} f(q^{2k}) = f(0)$
  2. Invertible
     - $f(q^{2k}) \neq 0,\ \forall\, k \geq 0$
     - $f(0) \neq 0$

$\overline{\delta}$-operator

- Recall that we have
$$\overline{\partial}(B_0^n) = \frac{1-q^{2n}}{1-q^2} B_0^{n-1} \overline{\partial}(B_0)$$
- When looking at $\overline{\partial}$ restricted to $C^*\{1, B_0\}$, this introduces the following operator,

Definition

Let $\mathbb{C}[x] \subset C(sp(B_0))$ be the ring of polynomial functions. We define the operator $\overline{\delta}$ as:
$$\overline{\delta}(x^n) = \frac{1-q^{2n}}{1-q^2} x^{n-1}$$
Where $\overline{\delta}(1) = 0$.

- $\overline{\delta}$ is not a derivation.

$\overline{\delta}$ Continued

- We generalize the previous formula to define
$$\overline{\delta}(f) = \frac{f(x) - f(q^2 x)}{x - q^2 x}$$
- $\overline{\delta}$ is a closed operator on $C(sp(B_0))$.
- In the case f is a polynomial the formula can be shown to match the formula on the prior slide.

Proposition ([GRM+25])

- *A function $f \in Dom(\overline{\delta})$ if and only if the following limit exists:*
$$\lim_{k \to \infty} \frac{f(q^{2k}) - f(q^{2k+2})}{q^{2k} - q^{2k+2}}$$
- *For any $f \in Dom(\overline{\partial}) \cap C^*\{1, B_0\}$ the following holds:*
$$\overline{\partial}(f) = \overline{\delta}(f)\overline{\partial}B_0$$

Solving Equation with $\overline{\delta}$.

- Since $\overline{\partial} f = \overline{\delta} f \overline{\partial} B_0$, for $f, g \in C^*\{1, B_0\}$, we can solve the equation $\overline{\delta} g = g \overline{\delta} f$ instead of $\overline{\partial} g = g \overline{\partial} f$.
- Explicitly the equation $\overline{\delta} g = g \overline{\delta} f$ is
$$\frac{g(x) - g(q^2 x)}{x - q^2 x} = g(x) \frac{f(x) - f(q^2 x)}{x - q^2 x}.$$
- The above equation yields the recursive formula
$$g(q^{2n}) = g(q^{2n-2})(1 - f(q^{2n-2}) + f(q^{2n})).$$

Proposition ([GRM+25])

A solution to $\overline{\delta} g = g \overline{\delta} f$ is given by

$$g(q^{2n}) = g(1) \prod_{k=1}^{n} (1 - f(q^{2k-2}) + f(q^{2k})).$$

Properties of Solution

- $f \in \mathrm{Dom}(\overline{\delta}) \Rightarrow g \in \mathrm{Dom}(\overline{\delta})$.
- g is invertible $\Leftrightarrow$ There is no $n \in \mathbb{N}$ such that $f(q^{2n}) - f(q^{2n+2}) = 1$.

Definition

We say f has a **defective spot** at $n \in \mathbb{N}$ if $f(q^{2n-2}) - f(q^{2n}) = 1$. Denote the set of detective spots of f by S_f.

The above argument leads to the following result:

Theorem ([GRM+25])

For $f \in C^*(1, B_0)$, $\overline{\nabla}_{\overline{\partial} f}$ is gauge equivalent to the standard antiholomorphic connection $\overline{\nabla}$ if and only if $S_f = \emptyset$.

Example

- $\frac{B_0}{1-q^2}$ has a defective spot at 1. In general,

$$\frac{B_0}{q^{2n-2}(1-q^2)}$$

has a defective spot at n.

- We have

$$\bar{\partial} B_0^\infty = B_0^\infty \frac{\bar{\partial} B_0}{1-q^2}$$

where $B_0^\infty = \lim_{m\to\infty} B_0^m$ is not invertible

- Therefore $\overline{\nabla}_{\frac{\bar{\partial}B_0}{1-q^2}}$ is not gauge equivalent to $\overline{\nabla}$.
- We found a concrete example of a $\bar{\partial}$-connection which is **not** gauge equivalent to the standard one.
- [Pol06]: Similar result for noncommutative two-tori.

Non-Gauge Equivalency

- We can further show that functions in $C^*\{1, B_0\}$ with the same defective spots are gauge equivalent.
- It is difficult to prove its inverse, hence we take a different approach to the non-gauge equivalent problem.
- We will focus on $\mathcal{L}_0$ from now on.
- Using some general facts on spectral triples, we can prove that $\ker \overline{\nabla}_\theta$ is **finite dimensional** for any $\overline{\nabla}_\theta$.
- Notice that $\dim \ker \overline{\nabla}_\theta$ is invariant under gauge equivalence.
- We will give a lower bound of $\dim \ker \overline{\nabla}_\theta$.

Considering $C^*\{1, B_0\}B_-^n$

- To find elements of $\ker(\overline{\nabla}_\theta)$, we consider elements of the form fB_-^n, for $f \in C^*\{1, B_0\}$ and $n \in \mathbb{N}$.
- For $\theta = \bar{\partial}h$, one can calculate that

$$\overline{\nabla}_\theta(fB_-^n) = \left(q^{2n}B_0\bar{\delta}f + (q^{n-1}[n] - q^{2n}(m_{q^{2n}}\bar{\delta}h)B_0)f\right)B_-^{n-1}\bar{\partial}B_-.$$

- Here, m represents the dilation operator: $(m_{q^{2n}}f)(x) = f(q^{2n}x)$.
- $fB_-^n \in \ker(\overline{\nabla}_\theta)$ yields the recursive formula

$$f(q^{2m}) = f(1)\prod_{k=0}^{m-1}\frac{1 - h(q^{2n+2k}) + h(q^{2n+2k+2})}{q^{2n}}.$$

- Since $q^{-2n} > 1$, it diverges as $m \to \infty$, **unless** there is a zero term in the product.

Infinitely many gauge equivalent classes

Proposition

If $n \leq \max S_h$ *then there exists a nonzero* $f \in C^*\{1, B_0\}$ *such that* $fB_-^n \in \ker \overline{\nabla}_{\bar{\partial}h}$.

- We can further prove that the fB_-^n's are linearly independent for different n.

Corollary

For $h \in C^*(1, B_0)$, *we have* $\dim(\ker(\overline{\nabla}_{\bar{\partial}h})) \geq \max\{S_h\}$.

- For $h = \frac{B_0}{q^{2n-2}(1-q^2)}$ we have $S_h = \{n\}$ hence $\dim(\ker(\overline{\nabla}_{\bar{\partial}h})) \geq n$.

Theorem ([GRM+25])

There exist at least countably many gauge equivalent classes of antiholomorphic connections on $\mathcal{L}_0$, *hence on any* $\mathcal{L}_n$ *for each* $n \in \mathbb{Z}$.

Possible further work

- Give a complete classification and parametrization of gauge equivalent classes of antiholomorphic connections on $\mathcal{L}_n$ (see [BGK52] for some recent progress in this direction).
- Explore non-standard complex structures on the "base complex manifold" $\mathbb{C}P^1_q$ rather than line bundles.
- Consider the same problem on higher dimensional $\mathbb{C}P^n_q$ or more general quantum Grassmannians.

Thank you!


Konrad Aguilar and Jens Kaad.

The Podleś sphere as a spectral metric space.

J. Geom. Phys., 133:260–278, 2018.


Indranil Biswas, Satyajit Guin, and Pradip Kumar.

Fixed points and holomorphic structures on line bundles over the quantum projective line $\mathbb{C}\mathrm{P}^1_q$, arXiv: 2601.06852.


Edwin Beggs and S. Paul Smith.

Non-commutative complex differential geometry.

J. Geom. Phys., 72:7–33, 2013.


Francesco D'Andrea and Giovanni Landi.

Geometry of quantum projective spaces.

In *Noncommutative geometry and physics. 3*, volume 1 of *Keio COE Lect. Ser. Math. Sci.*, pages 373–416. World Sci. Publ., Hackensack, NJ, 2013.


Mary Graveman, Landen La Rue, Lillian MacArthur, Hunter Pesin, and Zhaoting Wei.

Non-standard holomorphic structures on line bundles over the quantum projective line, arXiv:2510.14263, 2025.


Masoud Khalkhali, Giovanni Landi, and Walter Daniël van Suijlekom.

Holomorphic structures on the quantum projective line.

Int. Math. Res. Not. IMRN, (4):851–884, 2011.


Alexander Polishchuk.

Analogues of the exponential map associated with complex structures on noncommutative two-tori.

Pacific J. Math., 226(1):153–178, 2006.